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Examining Students' Mathematical Understanding of Geometric Transformations Using the Pirie-Kieren Model*

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Abstract

Students should learn mathematics with understanding. This is one of the ideas in the literature on mathematics education that everyone supports, from educational politicians to curriculum developers, from researchers to teachers, and from parents to students. In order to decide whether or not students understand mathematics we should first identify how mathematical understanding occurs. The purpose of this research is to analyze 10th-grade students' mathematical understanding of geometric transformations as developed in an environment enriched with multiple representations. Four 10th-grade students were observed during their lessons on translation, rotation, reflection, and dilation; semi-structured task-based interviews were then conducted with them after the lessons. The findings of this study reveal that although students' levels of mathematical understanding developed from informal to formal, this development was not unidirectional and students showed a tendency to use informal understandings. Students' primitive knowledge of geometric transformations was at the core of their understanding, whereas activities in the understanding levels of Image Making and Property Noticing directly affected the growth of their mathematical understanding. The folding back movements, activities in the forms of acting and expressing within the different levels of understanding, and multiple representations of concepts in the learning environment guided their process of mathematical understanding.

Keywords: Mathematical understanding • Pirie-Kieren Model • Geometric transformations • Representations • High school students

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The content of curriculum designs and innovations made in the field of mathematics education in the last decade have been shaped with emphasis on the need for students to have understanding while learning mathematics. For example, the recently updated curriculum for secondary mathematics in Turkey indicates that learning environments which "do not emphasize meaning or do not provide students with an opportunity or possibility to create meaning from the mathematics that is being learned" cannot properly meet the expectations of teaching (Ministry of Education [MoNE], 2013, p. 1). Similarly, according to the National Council of Teachers of Mathematics ([NCTM], 2000), "students must learn mathematics with understanding, actively building new knowledge from experience and prior knowledge," (p. 20). How do students then learn mathematics with understanding?

Hiebert and Carpenter (1992) remind us that the major goal of mathematics education is to ensure mathematical understanding in the learning environment, and they indicate that researchers should focus on finding the answer to the question of how and in what ways students make sense of mathematics. As "widespread rhetoric," mathematical understanding is vital for teachers, researchers, and curriculum developers (Mousley, 2005, p. 553). When the nature of mathematical understanding is characterized, researchers can focus on studies that analyze students' understandings in detail, curriculum developers can design the curriculum according to the growth of mathematical understanding, and teachers can organize their teaching goals based on this growth (MacCullough, 2007). Researchers have endeavored to identify the complex nature of mathematical understanding with respect to the learning theories which they support and the interpretations they have made of the term understanding (Meel, 2003). An important part of this effort includes the work done by researchers to describe the development of students' mathematical understanding via mental processes based on the constructivist approach.

One should consider the individual experiences, perceptions, and interactions students carry out in their environment when analyzing the process of mathematical understanding from the constructivist perspective (Pirie & Kieren, 1992). Many researchers have looked at learning from this perspective and provided distinct theories to characterize mathematical understanding (i.e., Davis, 1984; Sfard, 1991; Sierpinska, 1994; Skemp, 1978). One of these theories is the representation theory. This theory promotes the idea that students understand mathematics as much as they can make sense of the different representations of mathematical concepts and build connections among these representations (Goldin, 2003; Hiebert & Carpenter, 1992; Janvier, 1987). Goldin (2003) defines the term *representation* as a "configuration of signs, characters, icons, or objects that can somehow stand for, or 'represent' something else" (p. 276). Students work with representations of mathematical objects while dealing with mathematics across a variety of situations (Duval, 2006). They can learn mathematics with understanding if they can accurately apply different representations of a mathematical concept and construct the relationships between these representations (Lesh, Post, & Behr, 1987). Another theory that analyzes mathematical understanding from the constructivist perspective is the Pirie-Kieren theory, which was presented by Susan E. Pirie and Thomas Kieren in 1989. Pirie and Kieren (1994) define mathematical understanding as "a whole, dynamic, leveled but non-linear, transcendently recursive process" (p. 166). This theory emphasizes that the growth of mathematical understanding is not linear; by contrast, it improves dynamically with back and forth movements between mathematical ideas.

Mathematical Understanding Levels

Pirie and Kieren (1994) developed a model in order to represent ideas while they were building the theory of the growth of mathematical understanding (see Figure 1).

Figure 1: The model developed by Pirie and Kieren to represent the dynamic feature of mathematical understanding.

There are eight nested circles, which model the eight levels that can be encountered during the growth of mathematical understanding. The circles denote that each progressive level includes the previous levels of understanding, and mathematical understanding develops through back and forth movements within these circles. Although the model does not have a linear base, the regions between the circles are called levels because it has a certain hierarchy (Pirie & Kieren, 1994). The circles in the model show the levels of Primitive Knowing, Image Making, Image Having, Property Noticing, Formalising, Observing, Structuring, and Inventising from inner to outermost.

Primitive Knowing is knowledge about a topic to be explored, particularly a concept, that a student is assumed to be constructed previously (Pirie & Kieren, 1994). The circle that represents the second level is Image Making, an important and active process for the growth of understanding (Borgen, 2006). Students try to create an image of the concept by using their primitive knowledge during mental or physical activities at this level (Thom & Pirie, 2006). In the third circle of the Pirie-Kieren model, the level symbolizing Image Having, students have an image about the concept by means of the activities performed in the previous level. Independent of these activities, they "'know' some piece of mathematics" in regard to the concept (Thom & Pirie, 2006, p. 190). The next level is Property Noticing, where students question and use the different images they have developed. They examine the similarities and differences of their images and relate them to each other using particular mathematical statements (Thom & Pirie, 2006). At the level of Formalising, students generate a general statement about the concept by using these particular statements (Pirie & Kieren, 1994). They can construct the mathematical definition of the concept or develop formulas and algorithms about the topic (Borgen, 2006). As the sixth level of the Pirie-Kieren model, Observing is the level at which students observe the meaning they have formalized and organize their observations. They "reflect on and coordinate such formal activity and express such coordinations as theorems" at this level (Pirie & Kieren, 1994, p. 171). At the level of Structuring, students can capture a pattern by creating a synthesis of the observations they have made (Borgen, 2006). They can logically explain their formal observations, prove theorem-like expressions, and verify ideas that they had developed in the previous level (Thom & Pirie, 2006). The outermost level of the model is Inventising, where students look at their previously developed understanding, as Kieren (1992) says, in a "completely new way" and ask questions that lead them to invent totally new concepts (as cited in Borgen, 2006, p. 34). The four inner levels are described as the informal levels whereas the four outer levels are described as the formal levels of mathematical understanding (Pirie & Kieren, 1994).

The Characteristics of the Pirie-Kieren Theory

The Pirie-Kieren theory, in addition to the levels of understanding, indicates some important features called folding back, "don't need" boundaries, the complementary aspects of acting and expressing, and interventions. In the following section these features will be explained respectively.

Folding Back: Pirie and Kieren (1991) state that mathematical understanding proceeds with the help of folding back movements between levels where they define these movements as:

A person functioning at an outer level of understanding when challenged may invoke or fold back to inner, perhaps more specific local or intuitive understandings. This returned to inner level activity is not the same as the original activity at that level. It is now stimulated and guided by outer level knowing. The metaphor of folding back is intended to carry with it notions of superimposing ones current understanding on an earlier understanding, and the idea that understanding is somehow 'thicker' when inner levels are revisited (p. 172).

In other words, students fold back to the inner levels to broaden their existing insufficient understanding and reorganize previously constructed knowledge to develop new and appropriate images about the topic (Pirie, Martin, & Kieren, 1996).

"Don't Need" Boundaries: Some of the boundaries between the levels in the Pirie-Kieren model are shown with thicker lines (see Figure 1). These lines indicate an increasing abstract understanding (Borgen, 2006) and separate the model into four parts. A "don't need" boundary means that students no longer need the specific actions that have been carried out in the levels inside the boundary, and they can work with a more general and abstract level of understanding outside the boundary (Pirie & Kieren, 1994).

Complementary Aspects of Acting and Expressing: Pirie and Kieren (1994) classify activities experienced at the levels between Primitive Knowing and Inventising as acting and expressing. Acting is a mental or physical activity that "encompasses all previous understanding, providing continuity with inner levels," and expressing is "generally a verbal statement that gives distinct substance to that particular level," (p. 175).

Interventions: As a characteristic, interventions are "either internal or external" stimulating actions that lead students to review their present understanding (Borgen, 2006, p. 42). Pirie and Kieren (1994) categorize interventions as (*a*) provocative interventions that proceed the understanding to an outer level, (*b*) invocative interventions that cause folding back to an inner level, and (*c*) validating interventions that support and confirm the present level of work.

Researchers continue to use the Pirie-Kieren theory, which explains the complex and dynamic nature of the process of mathematical understanding with the levels of understanding and its features explained above. Martin (2008) noticed that the theory is still growing dynamically as it is used in different research areas such as teacher education, the growth of students' or pre-service teachers' understanding of mathematical concepts, the growth of students' mathematical understanding in computer-supported learning environments, the effect of student or teacher behaviors on mathematical understanding, and collective learning environments. Aside from these research areas, the theory has been elaborated on by Martin (1999), focusing on the feature of folding back, and by Towers (1998), focusing on the feature of teacher interventions in the learning environment.

Because the growth of mathematical understanding is a process rather than an acquired knowledge according to Pirie-Kieren theory, it is a purposive model to be used to describe the process of students' mathematical understanding and map their understanding as they engage different levels (Borgen, 2006; Martin, 2008; Warner, 2008). An analysis of research that has used this model (i.e., Martin, 2008; Nillas, 2010; Warner, 2008) suggests that research should continue to investigate the mathematical understanding of students using all components of the model. MacCullough (2007) also indicated that there is a need to conduct investigations that articulate the process of students' mathematical understanding of a concept rather than perform research that focuses on how students understand a concept, what they do not understand about a concept, or how they may develop a concept. To provide learning environments with understanding in schools it is important to characterize students' mathematical understanding after lessons in which they were exposed to multiple representations of mathematical concepts. In order to achieve this, researchers should obviously interact with students for a long period of time to examine their mathematical understanding in-depth. Therefore, in this research, four months was spent with four 10th-grade students in order to analyze their mathematical understanding of geometric transformations (henceforth described as just "transformations") in detail.

National curricula and standards (MoNE, 2013; NCTM, 2000), as well as quite a few researchers (i.e., Edwards, 2003; Flanagan, 2001; Hollebrands, 2003; Jung, 2002; Yanik, 2006), support the idea that transformations is a conceptual field that should be taught at schools in all grades. According to the National Council of Teachers of Mathematics ([NCTM], 2000), high school students "should understand and represent translations, reflections, rotations, and dilations of objects in the plane by using sketches, coordinates, vectors, function notation, and matrices" and they "should use various representations to help understand the effects of simple transformations and their compositions" (p. 397). Transformations offer students opportunities to work with important mathematical concepts such as functions and symmetry, to associate mathematics with other disciplines, and to reason by using different representations of mathematical concepts (Flanagan, 2001; Hollebrands, 2003). However, it is not easy for students to understand this important topic (Edwards, 2003; Flanagan, 2001; Hollebrands, 2003; Jung, 2002; Soon, 1989; Sünker & Zembat, 2012; Yanik, 2006, 2011; Yanik & Flores, 2009; Yavuzsoy-Köse, 2012). Research shows that students from all grades, even pre-service teachers, have difficulty dealing with transformations. Although a transformation on the plane is defined as a one-to-one correspondence from the plane onto itself (Dodge, 2012), students and pre-service teachers understand transformations as single motions on the plane (Edwards, 2003; Hollebrands, 2003; Yanik, 2006). The main difficulties students face in working with transformations are a) understanding the plane as the domain and range of transformations, b) confusing the parameters and the variables of transformations with each other, c) their informal experiences gained from using transformations in daily life, and d) an insufficient understanding of the fundamental mathematical concepts that have to be to be developed before transformations (Edwards, 2003; Hollebrands, 2003; Jung, 2002; Yanik, 2011). Because transformational geometry constitutes an important part of secondary mathematics curricula, more research is needed to articulate student understanding about transformations (Flanagan, 2001; Hollebrands, 2003; Soon, 1989). The purpose of this study is to analyze 10th-grade students' mathematical understandings of translations, rotations, reflections, and dilations which had been developed in a learning environment, enriched with multiple representations.

Methods

This study is based on the first author's ("the researcher" henceforward) doctoral dissertation, which was designed as a qualitative case study (Gulkilik, 2013).

The Research Context and Participants

The study was conducted in a high school located in one of the central districts of the capital city in Turkey. First, a six-week pilot study was carried out in three different 10th-grade classrooms of the school. The same mathematics teacher introduced several physical manipulatives regarding triangles and guided students in real world activities in each of these three classes. In the meantime, the researcher joined the classes during these six weeks and performed semi-structured participant observations of the context and students.

After six weeks, the research class was selected according to the students' reactions to the activities in the pilot study, their lesson attendance, and their motivation. The reason for not selecting one of the two classes was because students were usually passive during lessons. Also, many students from the other class which was not chosen were affiliated with social clubs who organized different activities during out-of-school-time. It was thought that these two conditions might negatively affect the study. The pilot study continued for four weeks in the research class. During this period, virtual manipulatives were introduced to the students and one hour of the two-hour geometry lessons was carried out by the teacher in the computer lab every week. By this means, it was aimed to help the teacher and students gain experience in the lessons performed with the support of manipulatives.

There were 32 students in the research class, 17 females and 15 males. Four students, two females and two males, were chosen as participants according to the maximum variation sampling (Patton, 2002). The researcher's classroom observations during the pilot study, teacher's opinions, the pretest administered to identify the pre-instructional knowledge about transformations, and spatial-ability test scores were taken into account during the selection of participants (see Table 1). The pretest consisted of open-ended questions prepared to determine students' abilities in using distinct representations of transformations and translating among these representations. The spatial ability test, implemented after the pretest, had two sub-dimensions, spatial orientation and spatial reasoning. Spatial ability was considered to be a criterion for selecting participants because it is an important component that affects the conceptual understanding of geometric concepts and problem-solving activities (Battista, 1990; Fennema & Tartre, 1985). Detailed information about these tools will be presented under the topic of data collection.

Table 1
Characteristics of Participants

Name	Age	Spatial Ability Test Score	Pretest Score	Geometry Achievement Score in Previous Semester
Defne	17	67.5	28	71.75
Elif	16	150.25	55	78.50
Metin	17	161.50	40	55
Selim	16	48	32	87

Note. Highest possible scores were 282 for Spatial Ability, 100 for the pretest, and 100 for geometry achievement.

The content of lessons for translation, rotation, reflection, and dilation were prepared during the pilot study. The lessons were designed with the help of two mathematics education professors and six pre-service secondary mathematics education teachers who were enrolled at a government university. The pre-service teachers applied the content design of each lesson onto their classmates in the Methods in Mathematics Education II course. The researcher and two mathematics education professors observed these applications. The researcher revised the lesson contents with the teacher of the research class after considering feedback from the two mathematics education professors. The teacher with fourteen years of experience was selected because she had positive attitudes regarding manipulatives and had used dynamic geometry software such as Geogebra and The Geometer's Sketchpad regularly during her geometry lessons.

The learning outcomes of secondary mathematics curriculum in Turkey propound that students should construct a formal understanding and make formal observations about transformations in 10th-grade. Therefore, lessons were prepared in a manner that students would be able to progress their understanding to the level of Observing. Physical and virtual manipulatives were used in addition to the verbal, graphical, and algebraic representations of transformations to enrich the lessons. Quite a few researchers highlighted the importance of using multiple representations in learning environments in order to strengthen the mathematical understanding of students (i.e. Goldin, 2003; Lesh et al., 1987; Ozgun-Koca, 1998). The physical manipulatives used during the instruction were designed with the pre-service teachers and revised by the authors according to feedback from two mathematics education professors (see Figure 2). Some basic criteria was considered during the design process such as (*a*) being based on the learning outcomes about transformations, (*b*) being dynamic, (*c*) being observed clearly in terms of its relation with other representations of the concept, and (*d*) being developed with easily accessible materials. After a broad search, http://www.mathsisfun.com, http://www.shodor.org, http://www.interaktifmatematik.com and http://nlvm.usu.edu were chosen as virtual manipulative websites, and two mathematics education professors were asked for their opinions about the convenience of manipulatives in these websites. The website for the National Library of Virtual Manipulatives (http://nlvm.usu.edu), designed by Utah State University, was selected for the study because the manipulatives on the website included all of the learning outcomes for translation, rotation, reflection, and dilation, and they were explicitly easy for students to use (see Figure 2).

After finishing the pilot study and all the preparations for the main study, lessons on translation, rotation, reflection, and dilation were performed by the teacher in the computer lab. Two lesson hours (2x50 minutes=100 minutes) per week were devoted for each of the transformations, therefore taking four weeks to complete all transformation lessons. All lessons were recorded with a video camera.

The learning outcomes for the transformations on which the lessons were based indicated that students should apply translation, rotation, reflection, and dilation, and they should identify and verify congruent and similar figures on plane. For this, the students used multiple representations of the transformations, sometimes during the teacher-guided activities and sometimes during the individual applications, to make sense of the related concepts. The teacher began the lessons by introducing manipulatives, visual and verbal representations, and then she presented the algebraic representations of the current transformation. She completed each lesson with particular problems in which distinct representations of the mathematical concepts were used. Hence, the students were expected to construct images for the transformations by using their previous knowledge, to examine the properties of transformations by using these images, to develop the formal meaning of the transformations by connecting these properties, and to make formal observations about the concepts with the help of multiple representations.

For example, the translation lesson began with a web-based video to draw the students' attention. Then the teacher wanted students to talk about the content of the video in order to learn students' images about the transformation. After sharing some examples of translations from daily life, students used virtual manipulatives to observe how the coordinates of a figure changed under a translation. The teacher introduced a physical manipulative of translation and discussed with the students the relationships between the original points and the image points of several geometric figures. She reminded students about the figures they had moved x-units to the right and left, and y-units up and down in middle-school transformation lessons and asked them whether or not they could determine these movements using vectors. After the teacher inserted the concept of vector into the lessons, the students continued using the physical manipulative to translate different geometric figures by different vectors. The teacher wanted them to use virtual manipulatives again to explore which properties remained invariant under a translation. For this purpose, she guided students in applications where students observed the differences between the original and image figures in terms of distance, angle measures, parallelism, and orientation; they then discussed their observations together.

After these applications, the mathematical definition of a transformation is introduced as a one-to-one correspondence that maps the points of plane onto plane, and the algebraic representation of translation was presented to share a common language with students. The teacher began to introduce algebraic representation with $T_{\vec{u}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$, where $\vec{u}$ is a vector on the plane. She denoted the image of the point P by P' where $P' = T_{\vec{u}}(P) = P + \vec{u}$ and clarified the related mathematical notations. She continued the lesson with applications where she expected students to realize differences when the translation vector was changed. She wanted students to use virtual manipulatives again to draw different figures on the plane, denoted as a rectangle on the screen, and translate one

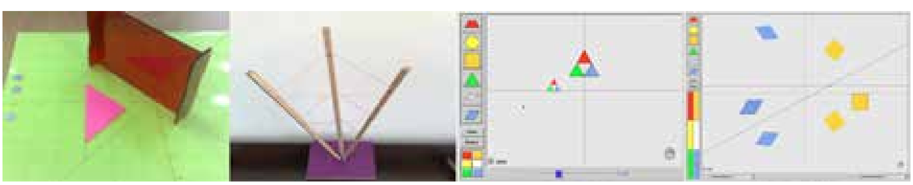

Figure 2: Samples from the physical and virtual manipulatives used during instruction.

of them by a vector. When students translated one of the figures, all of the other figures were also translated by the same vector; as a result, the teacher could emphasize that a translation transformed all points on the plane, not just the points of the present figure. She completed the lesson with exercises on GeoGebra and endeavored to link verbal, graphical, and algebraic representations to each other to strengthen students' understandings. The researcher joined in the lessons to perform semi-structured observations, focusing on the participants and taking field notes about their growth of mathematical understanding.

Data Collection Process

Before the transformation lessons a pretest was administered to all students in the research class to determine their pre-knowledge on translation, rotation, reflection, and dilation. The questions were prepared to examine students' previous understandings in terms of their ability to apply verbal, graphical, and algebraic representations of mathematical concepts in transformational geometry unit. The test was designed with the help of a mathematics education professor and included 26 open-ended questions about these four transformations. After the pretest, the spatial ability test, which was adopted from the tasks in the Kit of Factor-Referenced Cognitive Tests (Ekstrom, French, Harman, & Dermen, 1976) and translated into Turkish by Delialioğlu (1996), was given to the students to identify their spatial reasoning.

The first semi-structured interviews with the participants were conducted according to their responses to the pretest questions after the implementation of these two tools. The lessons began the following week with translation, and continued over the next three weeks with rotation, reflection, and dilation, respectively. Task-based semi-structured weekly interviews were carried out with participants after each transformation lesson to analyze the mathematical understanding they had developed during the introduction.

Task-based interviews were constructed to identify participants' mathematical application in one or more tasks (a question, problem, or any activity) in specific conditions that were predetermined by the researcher (Goldin, 2000). The semi-structured interview forms were prepared with the help of previous research studies on transformations (i.e., Jung, 2002; Soon, 1989; Yanik, 2006). The forms include open-ended questions that asked students to apply verbal, graphical, and algebraic representations of the current transformation and translate among these representations (see Appendix). The forms were finalized after considering the feedback of a mathematics education professor. Each of the weekly transformation interviews lasted an average of 50 minutes. The physical and virtual manipulatives used during instruction were made ready during the interviews. All of the interviews were video recorded and data was transcribed line-by-line to begin data analysis.

Data Analysis

Constant comparative method (Glaser & Strauss, 1967) was used to analyze the transcripts of the task-based semi-structured interviews. First, they were separated into parts and open-coded sentence by sentence. The codes were then evaluated carefully and relationships were constructed between the codes through axial coding. After making connections, the codes were put together under their related categories and themes were generated using category evaluation. The field notes taken during the participant observations were used to validate and support the interview data.

The gathered data was analyzed according to the Pirie-Kieren model, which provided the opportunity to observe the growth of mathematical understandings of students. A coding protocol was generated based on the components of the theory and the mathematical framework of the transformations. Protocol began with the codes and explanations for the theory's levels of mathematical understanding. Statements were provided to exemplify the possible activities of acting and expressing for each of the transformations that a student would perform in any level. The protocol continued with the codes and explanations about the two features of the theory: folding back and interventions. Features were clarified through sample statements of the transformations, thus determining the possible activities under each feature. Protocol was put into its final form by taking into account the feedback from two mathematics education professors. After completing the preparations, students' mathematical understandings of translation, rotation, reflection, and dilation were analyzed according to the protocol. Analysis was separately performed according to the levels of mathematical understanding and their features, and codes were then associated. For example, when a student was operating in the level of Formalising and needed to work on transformation properties using a manipulative, the statement was coded as *using manipulatives to fold back from Formalising to Property Noticing*. The Pirie-Kieren model with eight circles was used to explicitly present the growth of students' mathematical understanding to the reader. Serrated lines were used just as Pirie and Kieren (1994) did to show the extended work at a particular level.

In addition to triangulating the interviews with observations, prolonged engagement and persistent observations were used for ensuring the trustworthiness of the study. Thick description of the study and purposeful sampling were used for transferability, and member check through the application of Miles and Huberman's formula (1994) (agreements/[agreements + disagreements]) was used to verify credibility. An experienced mathematics education researcher with knowledge of the Pirie-Kieren theory but unaware of the context of the research analyzed 10% of the entire data. The researcher gave this mathematics education researcher the transformation data sets of two different participants and asked him to code the data according to the components of the Pirie-Kieren theory. The codes were compared and agreement was provided for 92% of the data. The remainder was discussed with him by making comparisons, and consensus was reached on the codes by the end of the discussion. Additionally, the findings were presented using direct quotes from the study, and documents related to the data were preserved.

Findings and Discussion

The findings of the study were formed around the traces students left at different levels of mathematical understanding while they were dealing with transformations. These traces were tracked by considering the folding back movements performed within the levels, the complementary activities in the form of acting and expressing from the levels of Image Making through Observing, and the interventions in the learning environment which affected the growth of mathematical understanding.

The Traces Left in Formal and Informal Understanding Levels

The lessons conducted throughout the study were designed based on the learning outcomes regarding transformations in the secondary school mathematics curriculum. According to these learning outcomes, students should be able to define translations, rotations, reflections, and dilations, as well as express formal observations including the specific properties that remain invariant under these transformations. However, the findings showed that even as students developed formal understandings during the lessons, they could not use these understandings while working on mathematical tasks during the interviews. In other words, although students had progressed their mathematical understanding to the level of Formalising or Observing, they were not able to use these levels of understanding independent of their inner levels of understanding. This situation reveals that the second "don't need" boundary was not an easy threshold to cross for students. Figure 3, which was used to model (*a*) Defne's understanding of translation, (*b*) Elif's understanding of rotation, (*c*) Selim's understanding of reflection, and (*d*) Metin's understanding of dilation, visually outlines this interpretation. The numbers on the figures are used to show the progression of students' understanding as they work on the tasks given during the interviews.

As seen in Figure 3, when Defne encountered translation tasks, she began to study using her previously constructed images. She then needed to construct new images and worked within the level of Image Making using her primitive knowledge. She folded back again to Primitive Knowing while working in the Formalising level. Although Elif preferred to use images to express her mathematical understanding about rotation, she worked within a formal level of understanding most of the time during the interview and folded back to Property Noticing because of her individual needs. Similarly, Selim described his images about the concept at the beginning of the reflection interview. Even though he was able to use his formal observation, he frequently folded back to Image Making to reorganize his images. When Metin met with the first task about dilation, he used his images and mostly worked within Formalising and Observing except for folding back to Primitive Knowing because of the algebraic representation of the transformation.

Table 2 presents an example showing how students' mathematical understanding was mapped according to their acting and expressing activities within levels. These mappings support the idea that students construct formal understandings from their informal understandings. These two understandings develop in a complementary manner and mathematical understanding grows as an integrated process (Pirie & Kieren, 1994).

Primitive Knowledge as the Core of Mathematical Understanding

Students' primitive knowledge of transformations as identified by the pretest before the lessons played

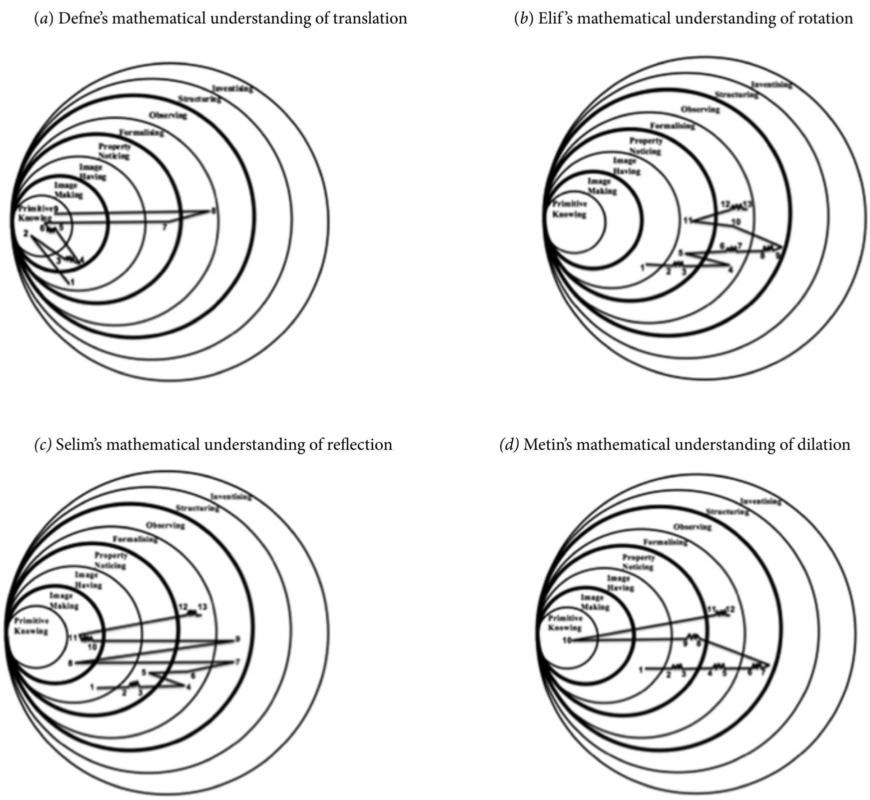

Figure 3: Mappings of participants' mathematical understanding for different transformations.

a key role in their growth of mathematical understanding. Primitive knowledge of transformations was found to be developed around the internal representations that students had constructed about basic geometric concepts such as vector, line, function, and plane that needed to be understood before transformations. By internal representation, we mean the verbal, imagistic, and formal-notational mental structures of a concept that a student creates by considering the mathematical or non-mathematical experiences in the learning environment (Goldin, 2003). Findings under this topic include

Table 2
Activities Elif Used While Dealing With Tasks During The Rotation Interview

Image Having	Property Noticing	Formalising	Observing
(1) "An example for rotation… Let me show here [using virtual manipulative], an object like that [drawing a triangle]. If we rotate this figure by 90º, it will be like that, by 180º it's like that, and by 270º like that. It will come to same place at by 360º [rotating the triangle by 90°, 180°, 270°, and 360° correctly in virtual manipulative]"	(2) "In rotation, the direction of the figure changes. In translation, it moves in the direction of the vector. I mean, because rotation is circular the figure moves in a circle." (3) "Rotating an object on the plane means rotating the object around a point that is not on the figure without changing the distance." (5) "First, I determine the distance between a point on the original figure [showing a triangle on virtual manipulative] and the rotational center. Then I mark the rotation angle and rotate the figure by this angle." (11) Elif determined the rotation angle as 270º clockwise and used the algebraic representations of the transformation to examine the properties of rotation.	(4) "This sentence, 'rotating a figure around a point by a specific angle on the plane' explains the rotation exactly." (6) Elif drew a rectangle in virtual manipulative, one corner of it was on the original point to be rotated, and the opposite point was on the rotation center. Then she found the image point by using this rectangle. (7) When Elif heard the term positive direction, she said that it meant "rotating the triangle in a counter-clockwise direction" and she wrote the mathematical formula of rotation while she indicated that she would "use the formula to find the image of a point under rotation if the rotation angle was other than 90º, 180º, 270º, or 360º". (10) Elif successfully found the rotation angle of a task in which the original figure and its' image after a rotation by 90º were given. (12,13) Elif was able to find the image of a triangle that was rotated by 60º around the origin. She could correctly apply the mathematical formula of the transformation and use both virtual and physical manipulatives effectively in this process.	(8) She stated "the original triangle and its' image were congruent after a rotation, only their location was different." (9) She stated that if the rotation angle was changed in a rotation, "the image of the figure would not change; only its location would change".

the implications of vector, ordered pair, line, function, image, one-to-one correspondence, and plane concepts which constitute a base for understanding transformations. As the students' understanding of these concepts was insufficient, they had some difficulties in developing a proper mathematical understanding about transformations. For example, it was found that students understood the term *display* from the term *image* when answering questions in which the properties of an image of a figure under a transformation were asked. The following sentences that Defne used when asked whether or not there would be any change with the image of an ABC triangle under a translation if the translation vector was changed exemplify this situation:

> Here, only the dimension changes. This distance (referring to the distance between the ABC triangle and its image). The image? Here, it does not change because they are similar. I mean, the ratios are the same; they are congruent.

Another incidence encountered during the following phase of the same interview indicates the role of basic concepts that students should previously have had an understanding of for appropriate mathematical understanding of transformations. Because Defne had not developed a proper understanding of the concept of vector, she had difficulties in understanding translations. When she was expected to draw the vector (1, 4) on the second quadrant, she stated that the components of the vector would change to (-1, 4) and it took a long time for her to overcome this problem.

Similarly, students' difficulties in understanding mathematical notations for the concept of plane and function as well as the parameters of transformations such as $\mathbb{R}^2 \rightarrow \mathbb{R}^2$, $T_{\vec{u}}(x, y)$, $R_\alpha(x,y)$ directly affected the growth of their mathematical understanding. The following sentences from Elif, who had developed a superior understanding among the participants, provide a good example of this finding. She explained what she understood from the expression $\mathbb{R}^2 \rightarrow \mathbb{R}^2$ in the following sentences:

> It is going from r square to r square... Like (2, 4). I mean, the first one, the x component, will be r squared, and the y component will be the square of x. Actually, I... This expression, from r square to r square is complicated ... It may also be (4, 16), similarly.

When one remembers that there may be many factors that influence students' primitive knowledge (Pirie & Kieren, 1994), the basic concepts that are required to understand transformations can be said to come before other factors. Students can develop a formal level of understanding to the extent that they have a proper and strong understanding of these primitive concepts. These findings are supported by other researchers who have examined the growth of mathematical understanding of different mathematical concepts (Grinevitch, 2004; Pirie & Kieren, 1994) and researchers who have studied students' understandings of transformations (Flanagan, 2001; Hollebrands, 2003; Soon, 1989; Yanik, 2011, 2013).

On the other hand, students' primitive knowledge about dilation was limited to stretching and shrinking activities on the plane because the dilation lesson in the study was the first time they had met with this transformation. Students' primitive knowledge was observed to also be a point of support for growth of mathematical understanding of transformations. Even as they began their explanations with a formal understanding, they used their previously constructed knowledge when needed. For example, Elif and Metin, who mostly worked within formal levels during their interviews, frequently needed to fold back to Primitive Knowing in order to study algebraic representations of translation and reflection. These findings are important for revealing the role of students' previously constructed knowledge regarding the current concept.

Images Determine the Path of Growth

Findings related to the growth of mathematical understanding during this process show the importance of the images that students constructed for transformations in addition to the core role of primitive knowledge. Participants had had experience with isometric transformations (translations, rotations, and reflections) up until they were in 10^{th}-grade, and had developed images based on the distinct external representations of these transformations, such as written expressions, graphics, pictures, diagrams, equations, and formulas. From this, the factors influencing students' images were determined during the pretest interview as (*a*) spoken language related to transformations in daily life, (*b*) mathematical and non-mathematical real-world experiences, and (*c*) the learning outcomes of transformations students having learned from middle school up to 9th grade.

Students' images about isometric transformations were discovered to be influenced by terms used for daily-life actions such as *replacing, moving, sliding, turning around,* and *superimposing.* For example, Defne explained translation as "moving a figure to the right or left, and/or up or down" while Elif explained it as "replacing a figure." The following dialogue which occurred between Elif and the researcher supports this finding:

Elif: Translation is changing the location of a figure without changing its direction, shape, area or size.

Researcher: How can I change its location?

Elif: Well, we cannot change its direction or area. We will only change its position.

Researcher: How can I change its position?

Elif: By sliding.

In addition to spoken language, real-world experiences were another factor that affected students' image development with isometric transformations. The following explanations that Metin made about these transformations, including also the above-mentioned daily-life actions, can be given as an example of this situation:

> Changing the position of a table is a translation. Turning a pencil around by fixing one of its points is a rotation. The fixed point is the origin; I mean the point (0, 0). Reflection... It is like seeing the below part of a folded, blank paper as the reflection of the above part of the same paper. The line in the middle of the paper is the axis of symmetry.

Students in the research class had previously been introduced to the three isometric transformations in middle school and to planar tessellations in the 9th grade. In the learning outcomes of those grades transformations were referred to as motions, and the students were required to explain and perform these motions (see MoNE, 2009, 2010). Participants' responses to the pretest interview questions showed that they had images related to these learning outcomes. The pretest-interview dialogue between Defne and the researcher can be used as evidence to support this finding.

Defne: For example, with translations from the x- or y-axis, we have the unit method, so many units right or so many units left. For example if we say right we make the translation on the positive x-axis, if we say left we make the translation on the negative x-axis. If it says up we make it on the positive y-axis and if it says down we make it on the negative y-axis. As an example, if it says translate the point A(x, y) 3 units right and 2 units down, it is (x +3, y−2). 3 units right on the x-axis and 2 units down on the y-axis.

Researcher: I got it. Is it like you drew here (see Figure 4)?

Defne: Yes, exactly.

Researcher: Ok. What can you say about rotation?

Defne: On the coordinate plane for example... We can turn a figure around by 90°, 180° or 270° in the first quadrant. After 360° it returns to its original position, becoming the same figure of course. It's like that.

Researcher: Did you think separately while you were drawing them?

Defne: Yes. For example, let's take this (shows the heart icon in the first quadrant). This is 90° (shows the heart icon in the fourth quadrant), this is 180° (shows the heart icon in the third quadrant) and this is 270° (shows the heart icon in the second quadrant).

Researcher: I see. Can you show in which direction you are turning it?

Defne: Yes, this direction (shows the clockwise direction).

Researcher: Ok. What can you say about reflection?

Defne: The axis of symmetry... Well, reflection means opposite. I mean in the mirror.

Students' images before the lessons could be said to be the starting point of growth in mathematical understanding. Students were observed to prefer beginning

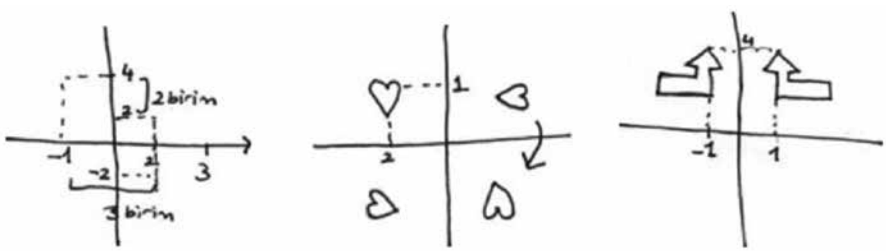

Figure 4: The drawings Defne made about translation, rotation, and reflection.

their explanations with these images during the weekly interviews for each isometric transformation. For example, Selim gave "ferris wheel," "a car wheel performs both translation and rotation," and "his image in the mirror," when asked to give examples of these transformations during the interviews conducted after the lessons. Similar findings were observed by Jung (2002), Flanagan (2001), and Yanik (2011), while studying transformations with students of different ages.

On the other hand, developing new proper images based on primitive knowledge was observed to have a big influence on the growth of mathematical understanding of transformations. At that point, mathematically inappropriate images were identified to cause trouble for students at further levels and took much time and effort to correct them. For example, Selim and Metin had difficulties with mathematical tasks because of the images that they had developed about algebraic notations for translation. They could not correct these images for a long time even while working on different examples of transformations. Both participants understood that the notation $T_{\vec{u}}(3, 4)$, where $\vec{u}$ = (1, -4), represents "the point (3, 4) as the image of a particular point translated by vector u". Selim's explanation of the notation precisely shows the images he had constructed while using physical manipulatives as follows:

> Our vector *u* is (1, -4), let me first draw it. (1, -4) is like that (draws the vector). It gave me its translated image, $T_{\vec{u}}(3, 4)$. Where is the point $T_{\vec{u}}(3, 4)$? Here (marks the point (3,4)). Well, if I choose any point as $P(x, y)$, I add the point $P(x, y)$ to the point (1, -4) and get the point (3,4). Then I think about it; what do I add to 1 to get 3? The x component is 2; what do I add to -4 to get 4? The y component is 8. Then I say that point P is the point (2, 8).

Although the teacher properly used this mathematical notation that included the images of different mathematical concepts such as function, domain, range, vector, and image, Selim and Metin had developed different images independent of the teacher's use of notations, and they preferred to use their own images in the learning environment. These findings effecting the growth of mathematical understanding revealed that knowledge is not something that can be easily transferred to students; on the contrary, it is something that students construct by themselves.

Image Making and Property Noticing according to Primitive Knowledge and Spatial Ability

Because students could not always use formal understanding, they worked mostly within the levels of Image Making and Property Noticing during the interviews conducted after their lessons (see Figure 3). Although the lessons were carried out to ensure that students could perform formal observations, students preferred to focus on the activities within these two levels. This observation suggests that students were continuing to develop the formalization process of mathematical understanding.

Students' primitive knowledge and spatial ability were the two main factors that determined the performance of students within Image Making and Property Noticing. As an example of the role of primitive knowledge, Metin had some difficulties with rotation while working on trigonometric functions during the interview; he needed to reorganize his images regarding algebraic representations of rotation at Image Making level. On the other hand, he did not encounter any problems regarding his primitive knowledge while working on reflection at a point; he preferred to apply some specific examples of reflection to observe properties of this transformation. The following sentences which come from his Image Making activities when asked to find the image of a triangle after a 90º rotation support this finding:

> Well, what was that? (writes $x\cos\alpha$ - $y\sin\alpha$, $x\cos\alpha$ + $y\sin\alpha$). I use that formula. The point (-2, 1), -2cos90 - 1sin90, this will give the first component. Let me put a comma (continues to write the previous formula). -2cos90 - 1sin90, -2cos90 + 1sin90. What were these? *cos*90, *sin*90? Let me draw a triangle (draws a right triangle and tries to calculate the value of *cos*90 and *sin*90). Adjacent edge divided by the hypotenuse, what was that...

The second factor affecting students' performance in these active levels can be said to be their spatial abilities. Elif and Metin, who were at an advantage in terms of spatial ability, mostly worked on activities using different properties of transformations. Defne and Selim, who were at a disadvantage in terms of spatial ability, usually worked on Image Making activities (see Table 1 and Figure 3). The researcher's observations of the participants during instruction verified this finding. Elif and Metin were faster at completing Image Making activities for transformations by using different representations, but Defne and Selim took more time and effort to apply these representations and develop images during the lessons. Primitive knowledge and spatial ability, two important factors, reveal that students should work within these two informal and active levels while trying to develop a strong mathematical understanding of transformations.

Folding Back Movements for Strengthening Mathematical Understanding

Students folded back to inner levels when they needed to study with more informal understandings even when they had been able to develop formal understandings (see Figure 3). Students made these movements to reorganize their insufficient primitive knowledge, to construct new or mathematically more acceptable images, and to work on different properties. Students' self-made decisions and the manipulatives in the learning environment were found to be the source of these movements, which have a very important role in the growth of mathematical understanding (Pirie & Kieren, 1994).

Students' folding back movements were in the form of needs to "work at inner layer using existing understanding" and to "collect at an inner layer" (Martin, 2008, p. 72). For example, Defne mostly studied with more informal understandings using her existing understanding and so frequently folded back to inner levels (see Figure 3). Similarly, because of the difficulties he had met during the mathematical tasks, Selim strengthened his understanding by folding back to inner levels and using manipulatives to collect at these levels (see Figure 5). For example, when his understanding was insufficient for quickly remembering his ideas that he provided at the Formalising level during the dilation interview, he knew what he needed and began to work on some of the properties of dilation as seen in the following sentences::

> First, I choose a center for dilation. Then, k times the distance between the center and the original point is equal to the distance between the center and the image point. It is like the example of Ataturk's tomb; you know there is a place with lions. If we choose this place as the center, let's say the distance is 100 meters between the center and the tomb of İsmet İnönü. In our scaled miniature this ration should be preserved. I mean, the distance was 100 meters, if I shrink it by 10, then the distance has to be 10 in my miniature.

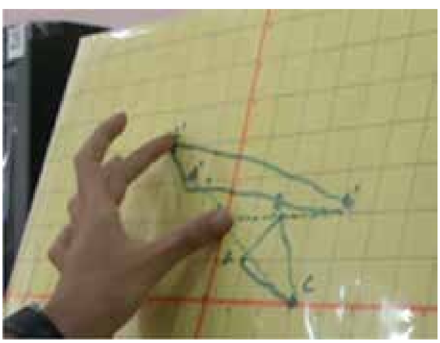

Figure 5: A situation in which Selim used his informal understandings on the physical manipulative.

The verbal, visual, and algebraic representations were observed to also help students strengthen their understanding by giving them the opportunity to reorganize their previously constructed understandings. For example, students folded back to as far as Primitive Knowing when they had to study algebraic representations of transformations. As an example, students had some difficulties trying to remember or use algebraic representations of reflections at a line even though they had been able to develop formal understandings. Hence, they applied their primitive knowledge and tried to reorganize their understanding of how to find an equation of a line whose two points were known or calculating the distance between a point and a line. The following dialogue between Elif and the researcher can be presented as an incident of this finding:

Elif: Figure 2 is the image of figure 1 after the reflection at a line. It is asking me to find the reflection line. Like that, this point (marks the point (-2, 3)) and this point (marks the point (1, 0)) (see Figure 6). Now, let's divide the distance between these two points into 2, because it has to be same with the line, I mean the distance between the points and the reflection line has to be same. I took one of them, and that one. 1, 2, 3 (calculating the length of the diagonals of unit squares between the points (–2,3) and the point (-2, 3)), here it is, one-and-half units. Then it will pass here (drawing the perpendicular bisector of the line segment between the points (–2,3) and (1, 0)). This is the reflection line.

Researcher: I see. Can you express this line algebraically? I mean can you find the equation of this line?

Elif: Let me try... What was the axis? It was *2M - P*. Let point M be on this line (marks the point *P′* as the point (-3, 1), the point *P′* as the point (-3, 1), the point P as the point (-1, -1), and finds the point M as the point (-2, 0) by using the equation *P′ = 2M - P*). We have to look for another point. point. Let's look for these (marks the point *P'* as the point (-2, 3), the point P as the point (1, 0), and finds the point *M* as (-1/2, 3/2) by using the equation *P′ = 2M - P*). The reflection line will pass both through (-2, 0) and (-1/2,3/2); we can find it by connecting them.

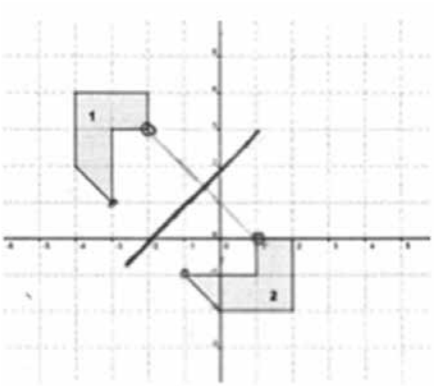

Figure 6: The drawings that Elif made while she was trying to find the reflection line.

Researcher: I see. Then can you find the equation of the line that passes through these two points?

Elif: Hmmm... The line where two points are given... I do not remember.

This finding reveals the importance of the experiences that students gain with distinct representations of mathematical concepts for developing strong mathematical understanding. Students got a chance to perform folding back movements to comprehend multiple representations and strengthened their mathematical understanding by revising and reorganizing their previous understandings. The results of the research conducted by Nillas (2010) and Wilson and Stein (2007) support these findings. They stated that students may need to work within the inner levels of mathematical understanding to build relationships among multiple representations of mathematical concepts.

The Critical Boundary, Acting-Expressing Aspects, and Interventions that Guide Mathematical Understanding

After folding back movements, findings point to the boundary between Property Noticing and Formalising, the second "don't need" boundary in the Pirie-Kieren model. Students performed activities independent of informal levels inside the boundary to the extent that they could work within the levels of Formalising and Observing. For example, Elif and Metin did not need to work on activities that were specific to different mathematical situations in the inner levels because they were able to use formal understandings of translations and dilations. However while they were studying rotations and reflections, they were dependent on their informal understandings because they could not use formal understanding with these transformations. This situation was also valid for Defne and Selim for all transformations. In this context, another important characteristic of the theory became prominent: the complementary aspects of acting and expressing. Students performed physical and mental activities as needed, expressing their abilities with these activities during the development of mathematical understanding. When the aforementioned critical boundary is considered, because students performed actions and in particular provided expressions within informal levels, they were disposed to using these kinds of understandings. In other words, although students had developed formal understandings about transformations, they could not always use these understandings because they could not find enough time to express the activities within formal levels.

Moreover, students used different representations of transformations including manipulatives during acting activities to ensure continuity and during expressing activities to provide strength of mathematical understanding. This situation highlights the importance of manipulatives and other multiple representations once more because students improve their mathematical understandings using these acting and expressing activities complementarily. These representations help students perform these actions and therefore progress their mathematical understanding to further levels.

Internal self-interventions and external interventions derived from environmental stimulants affected students' mathematical understanding while working on transformational tasks. Students' mathematical understandings were observed to be influenced by the manipulatives in their environment and the tasks that were prepared with verbal, graphical, and algebraic representations. Manipulatives and verbal/graphical representations of mathematical concepts intervened in their process of understanding in a provocative, validating, or invocative manner; algebraic representations intervened in mostly in an invocative manner. The following sentences that Selim used while working on one of the physical manipulatives of reflection can be given as an example of how manipulatives as invocative interventions influence student understanding (see Figure 7).

> Here, it is very important that the distances should be equal (shows the distance between the original point and the reflection line, and the distance between the image point and the reflection line). We know that there must be a reflection axis. Let's say the line $x = 0$, the axis, is the reflection axis for the coordinate system. We can easily see the image

if we put a mirror (the reflection mirror) on the reflection axis.

Figure 7: Selim uses the reflection mirror while expressing his ideas within the present level of understanding.

This situation shows that mathematical understanding as an integrated and dynamic process is affected by manipulatives, which is one of the multiple representations of a mathematical concept. Students cannot be said to be independent from the learning context; on the contrary, they are influenced by the experiences they have gained from the environment when trying to develop mathematical understanding of a concept.

Conclusions and Implications

The Pirie-Kieren model is a valuable tool that provides an opportunity to observe the process of students' mathematical understanding and present these observations in an organized manner. The first results of the study pertain to the path of growth of mathematical understanding progressing through the levels of understanding. Students' mappings of mathematical understanding of transformations show that they developed formal understandings based on their informal understandings. Therefore, teachers should not limit mathematical understanding only to the use of formal mathematical definitions or formulas; they should consider that students cannot construct formal understanding of a mathematical concept without the support of informal understandings on this concept. Mathematical understanding is not a static acquisition that can be gained by memorizing mathematical knowledge on boards or in books; it is a dynamic process which develops from informal to formal levels by building on some basic previous knowledge (Pirie & Kieren, 1994). Results reveal that expecting students to have formal mathematical understanding without experiencing enough informal understanding is against the nature of the process of mathematical understanding. Hence, teachers should provide learning environments that focus on all levels of mathematical understanding, from Primitive Knowing to Inventising. Additionally, students use informal and formal understandings complementarily while developing their mathematical understanding of a concept. In other words, even if they have formal understanding of the current topic, they may not always use this formal understanding. When they have a problem that they cannot overcome with formal understanding, they fold back and apply their informal understandings to strengthen the current mathematical understanding.

Further results related to the growth of mathematical understanding are about primitive knowledge and student images of transformations. Students' primitive knowledge was found to be composed of internal representations of some basic mathematical concepts such as vector, line, function, and plane that students should construct before they work with transformations. Teachers can try to identify student representations of these fundamental concepts as a preliminary step to teaching transformations. In addition to primitive knowledge, student images were affected by the language spoken in daily life, the mathematical and non-mathematical real-world examples, as well as the experiences they had in lessons about transformations in middle school. It is important for students' performance in the higher levels of mathematical understanding to build mathematically appropriate images based on their primitive knowledge. When they encounter a problem due to their images about a concept, they need plenty of time and effort to revise these images. Making the required preparations after determining students' primitive knowledge, including understanding the concepts of vectors, planes, ordered pairs, functions, and distance, as well as identifying students' images about transformations, can be said to simplify the work of teachers during transformation lessons.

Another recommendation that can be made to teachers regards the implementations that students perform within the levels of Image Making and Property Noticing. Students need to work within these levels more than other levels while developing mathematical understanding of transformations. This shows that they had to engage in activities in which they are physically and mentally active. Primitive knowledge and spatial ability were two determinants students used to decide the level they mainly work in. Students who have the advantage of strong primitive knowledge and spatial ability can be said to frequently study within the level of Property Noticing, whereas students who are disadvantaged in terms of these two factors can be said to usually engage in activities within the level of Image Making. Therefore, teachers should spend more time and effort helping students who need to improve primitive knowledge or spatial ability develop a strong mathematical understanding about transformations.

Students make folding back movements either because of a problem they cannot overcome with their present understanding or because of a need they feel to express their more informal understandings in support of formal understanding. In both situations, these movements directly influence their growth of mathematical understanding. Encouraging students to fold back to the inner levels in their learning environment is recommended considering the benefits these movements have while students try to build their understanding of mathematical concepts. In addition to folding back, acting and expressing activities within the levels of understanding are important for the growth of mathematical understanding. Results show that it takes time to pass the critical "don't need" boundary between the levels of Property Noticing and Formalising. Students preferred to study inside the boundary although they had developed formal understandings of transformations by using several representations and examining different properties of transformations. It may be because students are not experienced in the levels of Formalising and Observing. Students may need to work on theorems or theorem-like statements about transformations in order to use formal understandings as often as they use informal understandings. In other words, although the lessons are implemented in a design where students can progress their understanding to the levels of Formalising and Observing, it is not easy for them to build a formal understanding that can be used routinely. To build this formal understanding, students may be required to work on activities with more abstract and advanced mathematical tasks, as well as to express these activities so as to strengthen their level of mathematical understanding regarding Formalising and Observing using folding back movements. Students take their first steps in developing mathematical understanding with acting activities and they support and strengthen their understanding with expressing activities within a new level (Pirie & Kieren, 1994). Hence, teachers should provide learning environments that offer opportunities for students to express their mathematical understandings within the informal and formal levels of understanding, as well as perform physical and mental activities while they make back and forth movements between the levels of understanding.

Furthermore, students' mathematical understandings were observed to be affected by external and internal stimulants. These interventions are sometimes provocative as they help students reach further levels, sometimes invocative as they make students fold back to inner levels, and sometimes validating as they confirm students' present understanding. All of the components in the learning environment, from learning tools to the relations among different representations of the same concept, should be emphasized as potential interventions that may influence the growth of mathematical understanding. Therefore, teachers should be aware that every implementation they perform for a current topic may change the direction of the growth of students' mathematical understanding. According to the results, the main interventions were the manipulatives as well as the verbal, visual, and algebraic representations. These are external stimulants in the learning environment enriched with multiple representations of mathematical concepts. Teachers should apply and encourage students to work with these representations at all levels of mathematical understanding because students use these representations when they fold back between Primitive Knowing and Observing, performing acting and expressing activities within these levels. In this way, multiple representations help students to strengthen their mathematical understanding and progress to further levels.

Aside from educators, there are some main recommendations for researchers who want to study students' mathematical understanding of transformations or any other mathematical concept. For example, the results of this study reveal that students' primitive knowledge of transformations differs from each other. Students' primitive knowledge may be analyzed in detail, and mappings of students' growth of mathematical understanding may be compared within the context of these differences. Also, the focus may be directed to the acting and expressing activities that students perform within the levels of Image Making and Property Noticing. Because students are active in these levels more than other levels and the activities they perform in these levels give direction to their process of mathematical understanding. Manipulatives and other representations of transformations in the environment were determined to influence students' mathematical understanding. Therefore, studies that examine the role of different representations as external stimulants (especially physical and virtual manipulatives) for the growth of mathematical understanding may provide important contributions to the literature on mathematical understanding.

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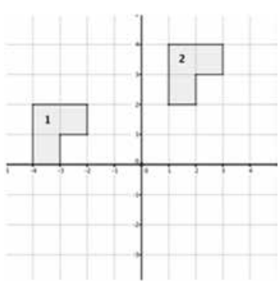
Appendix

Samples of the Weekly Interview Questions about Translation

1) Triangle ABC has vertices, $A(-3,-5)$, $B(-3,-2)$, and $C(1,-5)$. Can you find the image of the triangle ABC under translation by vector $\vec{u} = (1, 4)$? Explain your ideas and justifications in terms of the things that you noticed while finding the image.

- Can you see any relationship between the original triangle and its image? Explain your ideas.
- What do you think would happen to the image of triangle ABC if you change the length of the vector? Why?

2) Look at the following image. Figure 2 is the image of figure 1 under a translation. Can you find the translation vector? Explain your ideas and justifications in terms of the things that you noticed while finding the vector.



3) $T_{\vec{u}}: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ and $T_{\vec{u}}(P) = P + \vec{u}$ where $\vec{u} = (1, 4)$. Can you find the location of the point $T_{\vec{u}}(3, 4)$? Explain your ideas and justifications in terms of the things that you noticed while finding the location.